SADLER MATHEMATICS SPECIALIST UNIT 2

WORKED SOLUTIONS

Chapter 13 Complex numbers

Exercise 13A

Question 1

$\sqrt{-25} = 5i$

Question 2

$\sqrt{-144} = 12i$

Question 3

$\sqrt{-9} = 3i$

Question 4

$\sqrt{-49} = 7i$

Question 5

$\sqrt{-400} = 20i$

Question 6

$\sqrt{-5} = \sqrt{5}i$

Question 7

$\sqrt{-8} = 2\sqrt{2}i$

Question 8

$\sqrt{-45} = 3\sqrt{5}i$

Question 9

a 3

b 5

Question 10

a –2

b 7

Question 11

a 3

b –1

Question 12

$$\frac{-2\pm\sqrt{2^2-4\times1\times5}}{2\times1}$$
$$=\frac{-2\pm\sqrt{-16}}{2}$$
$$=\frac{-2\pm4i}{2}$$
$$=-1+2i,-1-2i$$

Question 13

$$\frac{-2\pm\sqrt{2^2-4\times1\times3}}{2\times1}$$
$$=\frac{-2\pm\sqrt{-8}}{2}$$
$$=\frac{-2\pm2\sqrt{2}i}{2}$$
$$=-1+\sqrt{2}i,-1-\sqrt{2}i$$

Question 14

$$\frac{-4\pm\sqrt{4^2-4\times1\times6}}{2\times1}$$
$$=\frac{-4\pm\sqrt{-8}}{2}$$
$$=\frac{-4\pm2\sqrt{2}i}{2}$$
$$=-2+\sqrt{2}i,-2-\sqrt{2}i$$

Question 15

$$\frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 10}}{2 \times 1}$$
$$= \frac{-2 \pm \sqrt{-36}}{2}$$
$$= \frac{-2 \pm 6i}{2}$$
$$= -1 + 3i, -1 - 3i$$

Question 16

$$\frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 6}}{2 \times 1}$$
$$= \frac{4 \pm \sqrt{-8}}{2}$$
$$= \frac{4 \pm 2\sqrt{2}i}{2}$$
$$= 2 + \sqrt{2}i, 2 - \sqrt{2}i$$

Question 17

$$\frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times 1}}{2 \times 2}$$
$$= \frac{1 \pm \sqrt{-7}}{4}$$
$$= \frac{1}{4} + \frac{\sqrt{7}}{4}i, \frac{1}{4} - \frac{\sqrt{7}}{4}i$$

Question 18

$$\frac{-1\pm\sqrt{1^2-4\times2\times1}}{2\times2}$$

$$=\frac{-1\pm\sqrt{-7}}{4}$$

$$=-\frac{1}{4}+\frac{\sqrt{7}}{4}i,-\frac{1}{4}-\frac{\sqrt{7}}{4}i$$

Question 19

$$\frac{-6\pm\sqrt{6^2-4\times2\times5}}{2\times2}$$

$$=\frac{-6\pm\sqrt{-4}}{4}$$

$$=\frac{-6\pm2i}{4}$$

$$=-\frac{3}{2}+\frac{1}{2}i,-\frac{3}{4}-\frac{1}{2}i$$

Question 20

$$\frac{2\pm\sqrt{(-2)^2-4\times2\times25}}{2\times2}$$

$$=\frac{2\pm\sqrt{-196}}{4}$$

$$=\frac{2\pm14i}{4}$$

$$=\frac{1}{2}+\frac{7}{2}i,\frac{1}{2}-\frac{7}{2}i$$

Question 21

$$\frac{2\pm\sqrt{(-2)^2-4\times5\times13}}{2\times5}$$
$$=\frac{2\pm\sqrt{-256}}{10}$$
$$=\frac{2\pm16i}{10}$$
$$=\frac{1}{5}+\frac{8}{5}i,\frac{1}{5}-\frac{8}{5}i$$

Question 22

$$\frac{1\pm\sqrt{(-1)^2-4\times1\times1}}{2\times1}$$
$$=\frac{1\pm\sqrt{-3}}{2}$$
$$=\frac{1\pm\sqrt{3}i}{2}$$
$$=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

Question 23

$$\frac{3\pm\sqrt{(-3)^2-4\times5\times1}}{2\times5}$$
$$=\frac{3\pm\sqrt{-11}}{10}$$
$$=\frac{3\pm\sqrt{11}i}{10}$$
$$=\frac{3}{10}+\frac{\sqrt{11}}{10}i,\frac{3}{10}-\frac{\sqrt{11}}{10}i$$

Exercise 13B

Question 1

$7+2i$

Question 2

$3-10i$

Question 3

$-3+4i$

Question 4

$7-2i$

Question 5

$-3+2i$

Question 6

$7-2i$

Question 7

$13+4i$

Question 8

$6+4i+6+3i$
$=12+7i$

Question 9

$10+5i+3-3i$
$=13+2i$

Question 10

$10+5i-3+3i$
$=7+8i$

Question 11

$3-15i+7i$
$=3-8i$

Question 12

$3-15i+7$
$=10-15i$

Question 13

$2+5=7$

Question 14

$4+1=5$

Question 15

$(3+2i)(2+5i)$
$= 6+15i+4i+10i^2$
$= 6+19i-10$
$= -4+19i$

Question 16

$(1+3i)(3+2i)$
$= 3+2i+9i+6i^2$
$= 3+11i-6$
$= -3+11i$

Question 17

$(2+i)(1-i)$
$= 2-2i+i-i^2$
$= 2-i+1$
$= 3-i$

Question 18

$(-2+3i)(5+i)$
$= -10-2i+15i+3i^2$
$= -10+13i-3$
$= -13+13i$

Question 19

$$\frac{(3+2i)}{(1+5i)}\times\frac{(1-5i)}{(1-5i)}$$

$$=\frac{3-15i+2i-10i^2}{1-25i^2}$$

$$=\frac{3-13i+10}{26}$$

$$=\frac{13-13i}{26}$$

$$=\frac{1}{2}-\frac{1}{2}i$$

Question 20

$$\frac{(3+i)}{(1-2i)}\times\frac{(1+2i)}{(1+2i)}$$

$$=\frac{3+6i+i+2i^2}{1-4i^2}$$

$$=\frac{3+7i-2}{5}$$

$$=\frac{1+7i}{5}$$

$$=\frac{1}{5}+\frac{7}{5}i$$

Question 21

$$\frac{4}{(1+3i)}\times\frac{(1-3i)}{(1-3i)}$$

$$=\frac{4-12i}{1-9i^2}$$

$$=\frac{4-12i}{10}$$

$$=\frac{2}{5}-\frac{6}{5}i$$

Question 22

$$\frac{2i}{(1+4i)}\times\frac{(1-4i)}{(1-4i)}$$

$$=\frac{2i-8i^2}{1-16i^2}$$

$$=\frac{2i+8}{17}$$

$$=\frac{8}{17}+\frac{2i}{17}$$

Question 23

$$\frac{(-3+2i)}{(2+3i)}\times\frac{(2-3i)}{(2-3i)}$$

$$=\frac{-6+9i+4i-6i^2}{4-9i^2}$$

$$=\frac{-6+13i+6}{4+9}$$

$$=\frac{13i}{13}$$

$$=i$$

Question 24

$$\frac{(5+i)}{(2i+3)}\times\frac{(2i-3)}{(2i-3)}$$

$$=\frac{10i-15+2i^2-3i}{4i^2-9}$$

$$=\frac{-15+7i-2}{-4-9}$$

$$=\frac{-17+7i}{-13}$$

$$=\frac{17}{13}-\frac{7}{13}i$$

Question 25

a $9+i$

b $(5-2i)-(4+3i)$

$=1-5i$

c $3(5-2i)-2(4+3i)$

$=15-6i-8-6i$

$=7-12i$

d $(5-2i)(4+3i)$

$=20+15i-8i-6i^2$

$=20+7i+6$

$=26+7i$

e $(4+3i)^2$

$=16+24i+9i^2$

$=16+24i-9$

$=7+24i$

f $\dfrac{(5-2i)}{(4+3i)}\times\dfrac{(4-3i)}{(4-3i)}$

$=\dfrac{20-8i-15i+6i^2}{16-9i^2}$

$=\dfrac{20-23i-6}{16+9}$

$=\dfrac{14-23i}{25}$

$=\dfrac{14}{25}-\dfrac{23}{25}i$

Question 26

a 4

b $(1-5i)-(3+5i)$

$=-2-10i$

c $(3+5i)+3(1-5i)$

$=3+5i+3-15i$

$=6-10i$

d $(3+5i)(1-5i)$

$=3-15i+5i-25i^2$

$=3-10i+25$

$=28-10i$

e $(3+5i)^2$

$=9+30i+25i^2$

$=9+30i-25$

$=-16+30i$

f $\frac{(3+5i)}{(1-5i)}\times\frac{(1+5i)}{(1+5i)}$

$=\frac{3+5i+15i+25i^2}{1-25i^2}$

$=\frac{-22+20i}{26}$

$=-\frac{11}{13}+\frac{10}{13}i$

Question 27

a $24+7i$

b $24-7i+(24+7i)$

$=48$

c $(24-7i)(24+7i)$

$=576-49i^2$

$=576+49$

$=625$

d $\frac{(24-7i)}{(24+7i)}\times\frac{(24-7i)}{(24-7i)}$

$=\frac{576-336i+49i^2}{576-49i^2}$

$=\frac{576-336i-49}{576+49}$

$=\frac{527-336i}{625}$

$=\frac{527}{625}-\frac{336}{625}i$

Question 28

a $4-9i$

b $4+9i-(4-9i)$

$=18i$

c $2(4+9i)+3(4-9i)$

$=8+18i+12-27i$

$=20-9i$

d $2(4+9i)-3(4-9i)$

$=8+18i-12+27i$

$=-4+45i$

e $(4+9i)(4-9i)$

$=16-81i^2$

$=16+81$

$=97$

f $\frac{(4+9i)}{(4-9i)}\times\frac{(4+9i)}{(4+9i)}$

$=\frac{16+72i+81i^2}{16-81i^2}$

$=\frac{16+72i-81}{16+81}$

$=\frac{-65+72i}{97}$

$=-\frac{65}{97}+\frac{72}{97}i$

Question 29

$z=w$

$2+ci=d+3i$

$d=2$

$c=3$

Question 30

$$a+bi=(2-3i)^2$$
$$=4-12i+9i^2$$
$$=4-12i-9$$
$$=-5-12i$$
$$a=-5, b=-12$$

Question 31

$$z=w$$
$$5-(c+3)i=d+1+7i$$
$$5=d+1$$
$$d=4$$
$$-(c+3)=7$$
$$c+3=-7$$
$$c=-10$$

Question 32

$$(a+3i)(5-i)=p$$
$$5a+15i-ai-3i^2=p$$
$$5a+3+(15-a)i=p$$
$$15-a=0$$
$$a=15$$
$$p=5a+3=5(15)+3=78$$

Question 33

a Statement is correct

If $z=a+bi, w=a-bi \therefore \text{Im}(w)=-\text{Im}(z)$

b Not a correct statement. The two complex numbers could have different real parts.

E.g. $w=5+3i, z=7-3i$ has $\text{Im}(z)=-\text{Im}(w)$ but w and z are not conjugates

Question 34

a

$$x^2 - 4x + 13 = 0$$
$$(x-2)^2 - 4 + 13 = 0$$
$$(x-2)^2 + 9 = 0$$
$$(x-2)^2 = -9$$
$$(x-2) = \pm\sqrt{-9}$$
$$x = 2 \pm 3i$$

$$x^2 - 4x + 13$$
$$= (x-(2+3i))(x-(2-3i))$$
$$= (x-2-3i)(x-2+3i)$$

b

$$x^2 - 2x + 10 = 0$$
$$(x-1)^2 + 9 = 0$$
$$(x-1)^2 = -9$$
$$x - 1 = \pm\sqrt{-9}$$
$$= \pm 3i$$
$$x = 1 \pm 3i$$

$$x^2 - 2x + 10$$
$$= (x-(1+3i))(x-(1-3i))$$
$$= (x-1-3i)(x-1+3i)$$

c

$$x^2 - 6x + 1 = 0$$
$$(x-3)^2 - 8 = 0$$
$$(x-3)^2 = 8$$
$$x - 3 = \pm 2\sqrt{2}$$
$$x = 3 \pm 2\sqrt{2}$$

$$x^2 - 6x + 1$$
$$= (x-(3+2\sqrt{2}))(x-(3-2\sqrt{2})$$
$$= (x-3-2\sqrt{2})(x-3+2\sqrt{2})$$

d $x^2 + 10x + 26 = 0$

$$(x+5)^2 + 1 = 0$$
$$(x+5)^2 = -1$$
$$x + 5 = \pm\sqrt{-1}$$
$$= \pm i$$
$$x = -5 \pm i$$

$$x^2 + 10x + 26$$
$$= (x - (-5+i))(x - (-5-i))$$
$$= (x + 5 - i)(x - 5 + i)$$

e $x^2 + 14x + 53 = 0$

$$(x+7)^2 + 4 = 0$$
$$(x+7)^2 = -4$$
$$x + 7 = \pm\sqrt{-4}$$
$$= \pm 2i$$
$$x = -7 \pm 2i$$

$$x^2 + 14x + 53$$
$$= (x - (-7-2i))(x - (-7+2i))$$
$$= (x + 7 + 2i)(x + 7 - 2i)$$

f $x^2 + 4x - 3 = 0$

$$(x+2)^2 - 7 = 0$$
$$(x+2)^2 = 7$$
$$x + 2 = \pm\sqrt{7}$$
$$x = -2 \pm \sqrt{7}$$

$$x^2 + 4x - 3$$
$$= (x - (-2+\sqrt{7}))(x - (-2-\sqrt{7}))$$
$$= (x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$$

Question 35

a Non real roots exist when the value of $b^2-4ac<0$.

Let $b^2-4ac<-m,$ then $\sqrt{b^2-4ac}=\sqrt{-m}=\pm\sqrt{m}i.$

The two solutions are then $\dfrac{x+\sqrt{m}i}{2a}$ and $\dfrac{x-\sqrt{m}i}{2a}$.

$\dfrac{x}{2a}+\dfrac{\sqrt{m}}{2a}i$ and $\dfrac{x}{2a}-\dfrac{\sqrt{m}}{2a}i$ are conjugates of each other.

b If one root is $3+2i$ the other root is $3-2i.$

$$\begin{aligned}x^2+bx+c&=(x-(3+2i))(x-(3-2i))\\&=(x-3-2i)(x-3+2i)\end{aligned}$$

The value of c is the product of the roots, $\begin{aligned}(3-2i)(3+2i)&=9-4i^2\\&=13\end{aligned}$

The value of b is the opposite of the sum of the roots.

$(3-2i)+(3+2i)=6$

$b=-6$

c The two roots are $5-3i$ and $5+3i.$

The product of the two roots is 34 and the sum is 10.

$c=34$ and $d=-10$

Question 36

a $\frac{c+di}{-c-di}$

$=\frac{c+di}{-(c+di)}$

$=-1$

$\frac{c+di}{d-ci}\times\frac{d+ci}{d+ci}$

$=\frac{cd+c^2i+d^2i+cdi^2}{d^2-c^2i^2}$

$=\frac{cd-cd+(c^2+d^2)i}{c^2+d^2}$

$=\frac{(c^2+d^2)i}{c^2+d^2}$

b $=i$

c $\frac{c-di}{-(d+ci)}\times\frac{(d-ci)}{(d-ci)}$

$=-\frac{cd-c^2i-d^2i+cdi^2}{d^2-c^2i^2}$

$=-\frac{cd-(c^2+d^2)i-cd}{c^2+d^2}$

$=-\frac{-(c^2+d^2)i}{c^2+d^2}$

$=i$

Question 37

$$\frac{3+5i}{1+pi}=q+4i$$

$$\begin{aligned}3+5i&=(q+4i)(1+pi)\\&=q+pqi+4i+4pi^2\\&=1-4p+(pq+4)i\end{aligned}$$

$$3=q-4p\Rightarrow q=3+4p$$

$$5=pq+4$$

$$5=p(3+4p)+4$$

$$\begin{aligned}0&=4p^2+3p-1\\&=(4p-1)(p+1)\end{aligned}$$

$$4p-1=0 \quad \text{or} \quad p+1=0$$

$$p=\frac{1}{4} \qquad\qquad p=-1$$

If $p=\frac{1}{4}, q=3+4\left(\frac{1}{4}\right)=4$

If $p=-1, q=3+4(-1)=-1$

Question 38

a $(x-z)(x-w)=x^2-(z+w)x+zw$

$a=1$

b From the above expansion, $z+w=b$ and we are told b is real.

We are also told c is real and $c=wz$.

c If $z+w$ is real, then Im(z)=0.

$z+w= p+qi+r+si$

$qi+si=0$

$q=-s$

Is also follows that

$zw=(p+qi)(r+si)$

$= pr+psi+qri+qsi^2$

$ps+qr=0$

$ps-sr=0$

$s(p-r)=0$

$p=r$ ($s \neq 0$ or our roots are not complex)

Our roots $p+qi$ and $r+si$ can now be expressed as $p+qi$ and $p-qi$ which are conjugates

Question 39

a $\overline{z} = a - bi$ and $\overline{w} = c - di$

$$\overline{z}\,\overline{w} = (a - bi)(c - di)$$
$$= ac - adi - bci + bdi^2$$
$$= ac - bd - (ad + bc)i$$

$$zw = (a + bi)(c + di)$$
$$= ac + bci + adi + bdi^2$$
$$= ac - bd + (bc + ad)i$$
$$\overline{zw} = ac - bd - (bc + ad)i$$

$$\Rightarrow \overline{z}\,\overline{w} = \overline{zw}$$

b $$\frac{z}{w} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$
$$= \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$
$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{ac + bd}{c^2 + d^2} - \frac{(bc - ad)i}{c^2 + d^2}$$

$$\frac{\overline{z}}{\overline{w}} = \frac{a - bi}{c - di} \times \frac{c + di}{c + di}$$
$$= \frac{ac + adi - bci - bdi^2}{c^2 - d^2 i^2}$$
$$= \frac{ac + bd}{c^2 + d^2} - \frac{bc + ad}{c^2 + d^2}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

Question 40

a (2, 3)

b (–5, 6)

c (0, 7)

d (3, 0)

e (1, 9)

f (6, 0)

g (3, 3)

h (0, 14)

i
$$\begin{aligned}&(0+2i)(3+5i)\\&=6i+10i^2\\&=6i-10\end{aligned}$$

(–10, 6)

j
$$\begin{aligned}&(-3+i)(-3-i)\\&=9+3i-3i-i^2\\&=10\end{aligned}$$

(10, 0)

k
$$\begin{aligned}&\frac{3+0i}{2-4i}\times\frac{2+4i}{2+4i}\\&=\frac{6+12i}{4-16i^2}\\&=\frac{6+12i}{20}\\&=\frac{3}{10}+\frac{3}{5}i\end{aligned}$$

$(\frac{3}{10},\frac{3}{5})$

l

$$\frac{3-8i}{3+8i}\times\frac{3-8i}{3-8i}$$
$$=\frac{9-48i+64i^2}{9-64i^2}$$
$$=\frac{-55-48i}{73}$$
$$=-\frac{55}{73}-\frac{48}{73}i$$

$$\left(-\frac{55}{73},-\frac{48}{73}\right)$$

Question 41

$$\frac{1}{z}=\frac{2+7i}{1-i}\times\frac{1+i}{1+i}$$
$$=\frac{2+2i+7i+7i^2}{1-i^2}$$
$$=\frac{-5+9i}{2}$$
$$z=\frac{2}{-5+9i}\times\frac{-5-9i}{-5-9i}$$
$$=\frac{-10-18i}{25+45i-45i-81i^2}$$
$$=\frac{-10-18i}{106}$$
$$=-\frac{5}{53}-\frac{9}{53}i$$

Exercise 13C

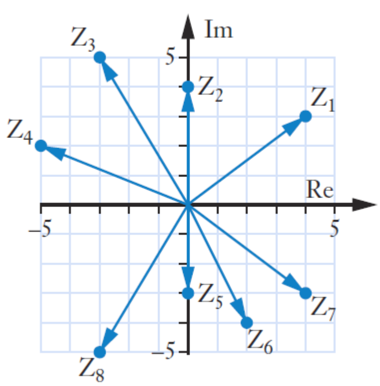
Question 1

$Z_1 = 7 + 2i$
$Z_2 = 2 + 4i$
$Z_3 = 0 + 6i$
$Z_4 = -5 + 3i$
$Z_5 = -7 - 5i$
$Z_6 = 0 - 4i$
$Z_7 = 3 - 6i$
$Z_8 = 6 - 3i$

Question 2

$Z_1 = (6, 0)$
$Z_2 = (7, 5)$
$Z_3 = (-3, 6)$
$Z_4 = (-5, 0)$
$Z_5 = (-6, -3)$
$Z_6 = (-3, -6)$
$Z_7 = (0, -6)$
$Z_8 = (7, -7)$

Question 3



Question 4

Z_3 and Z_2 are conjugates $\Rightarrow \text{Re}(Z_2) = \text{Re}(Z_3)$

$\dfrac{\text{Re}(Z_1)}{\text{Im}(Z_1)} > 0 \Rightarrow \text{Re}(Z_1)$ and $\text{Im}(Z_1)$ are both positive or both negative

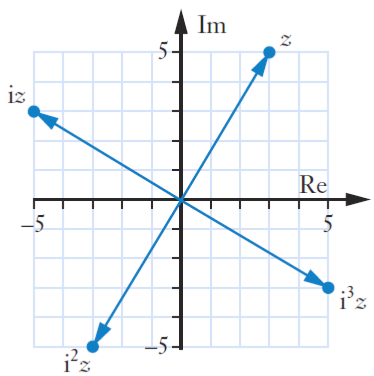
$Z_1 = 1 + 2i$

$Z_4 = 3 - 2i$

$\dfrac{\text{Re}(Z_2)}{\text{Im}(Z_2)} > 1 \Rightarrow Z_2 = -3 - 2i$

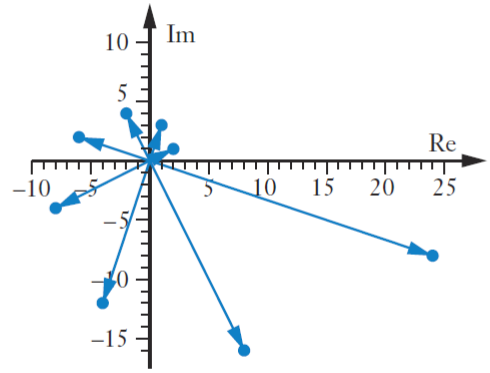
$Z_3 = -3 + 2i$

Question 5



Quetion 6

$$Z_2 = (2+i)(1+i)$$
$$= 2 + 2i + i + i^2$$
$$= 1 + 3i$$
$$Z_3 = (2+i)(1+i)^2$$
$$= (2+i)(1+2i+i^2)$$
$$= (2+i)(2i)$$
$$= 4i + 2i^2$$
$$= -2 + 4i$$



$$Z_4 = (2+i)(1+i)^3$$
$$= (2+i)(-2+2i)$$
$$= -6 + 2i$$

$$Z_5 = (2+i)(1+i)^4$$
$$= -8 - 4i$$

$$Z_6 = (2+i)(1+i)^5$$
$$= -4 - 12i$$

$$Z_7 = (2+i)(1+i)^6$$
$$= 8 - 16i$$

$$Z_8 = (2+i)(1+i)^7$$
$$= 24 - 8i$$

Question 1

a $(2+5i)(2-5i)$

$=4-25i^2$

$=29$

b $(3+i)(3-i)$

$=9-i^2$

$=10$

c $(6+2i)(6-2i)$

$=36-4i^2$

$=40$

d $(3+4i)(3+4i)$

$=9+24i+16i^2$

$=-7+24i$

e $\frac{2-3i}{3+i}\times\frac{3-i}{3-i}$

$=\frac{6-2i-9i+3i^2}{9-i^2}$

$=\frac{3-11i}{10}$

$=\frac{3}{10}-\frac{11i}{10}$

f $\frac{3+i}{2-3i}\times\frac{2+3i}{2+3i}$

$=\frac{6+2i+9i+3i^2}{4-9i^2}$

$=\frac{3+11i}{13}$

$=\frac{3}{13}+\frac{11i}{13}$

Question 2

a $-1+2i$

b $(2-3i)(-3+5i)$

$=-6+10i+9i-15i^2$

$=-6+10i+15$

$=9+19i$

c $2+3i$

d $9-19i$

e $(2-3i)(2-3i)$

$=4-12i+9i^2$

$=-5-12i$

f $(9+19i)(9+19i)$

$=81+342i+361i^2$

$=-280+342i$

g $\text{Re}(\overline{z})=2$

$\text{Im}(\overline{w})=-5$

$\Rightarrow p=2-5i$

Question 3

a $$(px-q)(x^2+rx+3) = px^3+prx^2+3px-qx^2-qrx-3q$$

$$= px^3+(pr-q)x^2+(3p-qr)x-3q$$

$$2x^3-5x^2+8x-3 = px^3+(pr-q)x^2+(3p-qr)x-3q$$

$p=2$

$-3q=-3 \Rightarrow q=1$

$3p-qr=8$

$3(2)-1r=8$

$r=-2$

b $2x^3-5x^2+8x-3=(2x-1)(x^2-2x+3)=0$

$2x-1=0$ or $x^2-2x+3=0$

$x=\frac{1}{2}$

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(3)}}{2}$$

$$=\frac{2\pm\sqrt{-8}}{2}$$

$$=\frac{2\pm2\sqrt{2}i}{2}$$

$$=1\pm\sqrt{2}i$$

$x=\frac{1}{2},1+\sqrt{2}i,1-\sqrt{2}i$

Question 4

a $2\times5=10$

b $\text{Re}\left((2+3i)(5-4i)\right)$

$=\text{Re}(10+7i-12i^2)$

$=\text{Re}(22+7i)$

$=22$

Question 5

a $-5\sqrt{2}i$

b $(5\sqrt{2}i)^2 = 25\times 2\times i^2$

$= 50i^2$

$= -50$

c $(1+5\sqrt{2}i)^2$

$= 1+10\sqrt{2}i+50i^2$

$= -49+10\sqrt{2}i$

Question 6

$(a+bi)^2 = 5-12i$

$a^2+2abi+b^2i^2$

$a^2-b^2+2abi = 5-12i$

$a^2-b^2 = 5$

$2ab = -12 \Rightarrow b = -\frac{12}{2a}$

$= -\frac{6}{a}$

$a^2-\left(-\frac{6}{a}\right)^2 = 5$

$a^2+\frac{36}{a^2} = 5$

$a^4+36 = 5a^2$

$a^4-5a^2+36 = 0$

$(a^2+4)(a-9) = 0$

$a^2 = -4$ $\quad a^2 = 9$

No such a $\quad a = \pm 3$

If $a = 3, b = -\frac{6}{3} = -2$

$a = -3, b = -\frac{6}{(-3)} = 2$

Question 7

a $\begin{bmatrix}1 & 3\\1 & 0\end{bmatrix}\begin{bmatrix}a & c & e\\b & d & f\end{bmatrix}=\begin{bmatrix}-1 & 10 & -4\\2 & -2 & -4\end{bmatrix}$

$$\begin{bmatrix}a & c & e\\b & d & f\end{bmatrix}=\begin{bmatrix}1 & 3\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}-1 & 10 & -4\\2 & -2 & -4\end{bmatrix}$$
$$=\frac{1}{3}\begin{bmatrix}0 & -3\\-1 & 1\end{bmatrix}\begin{bmatrix}-1 & 10 & -4\\2 & -2 & -4\end{bmatrix}$$
$$=\begin{bmatrix}2 & -2 & -4\\-1 & 4 & 0\end{bmatrix}$$

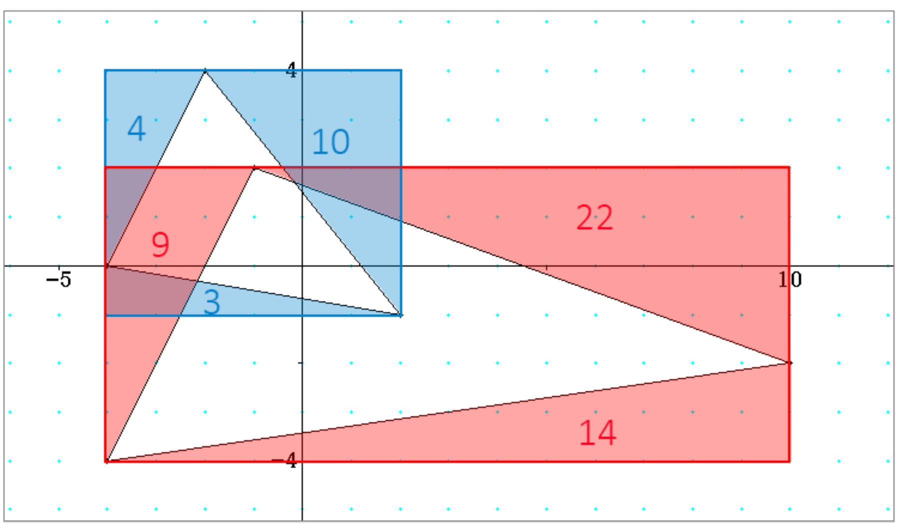
A (2, –1), B (–2, 4), C (–4, 0)

b Area of small triangle: $6\times 5-(3+4+10)=13$

Area of large triangle: $6\times 14-(9+14+22)=39$

$|\det T|=|1\times 0-1\times 3|=3$

Area $\Delta A'B'C'=|\det T|$ Area ΔABC



Question 8

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$a = 3, b = -1, c = 2, d = 1$$

Question 9

$$(z - 2 + 7i)^2 = -25$$

$$z - 2 + 7i = \pm 5i$$

$$z = 2 - 7i \pm 5i$$

$$z = 2 - 12i, 2 - 2i$$

Question 10

a Remembering the area of an image is the area of the original shape multiplied by the absolute value of the determinant of the transforming matrix, in this case our image has an area of zero which suggests the determinant of the 2 × 2 matrix must also be zero, making it singular.

b On page 218, we noted the origin is invariant under 2 × 2 transformations, meaning the origin is its own image. Therefore the line must pass through (0, 0).

Question 11

Let $z = a + bi,\ \overline{z} = a - bi$

$$z + 2z = 9 + 5i$$

$$a + bi + 2(a - bi) = 9 + 5i$$

$$3a - bi = 9 + 5i$$

$$3a = 9$$

$$a = 3$$

$$b = -5$$

Question 12

a $(2+3i)^4 = -119-120i$

b $(1-3i)^5 = 316+12i$

$\text{Im}\left((1-3i)^5\right) = 12$

Question 13

Method 1

$$\begin{aligned} x &= 2 \pm 3i \\ x-2 &= \pm 3i \\ (x-2)^2 &= -9 \\ (x-2)^2+9 &= 0 \\ x^2-4x+13 &= 0 \end{aligned}$$

Method 2

$$\begin{aligned} c &= (2+3i)(2-3i) \\ &= 4-9i^2 \\ &= 13 \\ -b &= (2+3i)+(2-3i) \\ &= 4 \\ b &= -4 \end{aligned}$$

$$x^2-4x+13=0$$

Question 14

$$\begin{aligned} 3(a+bi)+2(a-bi) &= 5+5i \\ 5a+bi &= 5+5i \\ 5a &= 5 \\ a &= 1 \\ b &= 5 \end{aligned}$$

Required number : $1+5i$

Question 15

$$(a+bi)(2-3i)=5+i$$
$$2a-3ai+2bi-3bi^2=5+i$$
$$2a+3b+(2b-3a)i=5+i$$
$$\left(2a+3b=5\right)\times 3 \Rightarrow 6a+9b=15-Eq1$$
$$\left(2b-3a=1\right)\times 2 \Rightarrow 4b-6a=2 \quad -Eq2$$
$$Eq1+Eq2$$
$$13b=17$$
$$b=\frac{17}{13}$$
$$2a=5-3b$$
$$=5-3\left(\frac{17}{13}\right)$$
$$a=\frac{7}{13}$$

$$z=\frac{7}{13}+\frac{17}{13}i$$

Question 16

$$z=-5+3i$$
$$z^2=(-5+3i)^2$$
$$=25-30i+9i^2$$
$$=16-30i$$

Question 17

$$\text{LHS} = \frac{\sin\theta}{\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$= 2\sin\left(\frac{\theta}{2}\right)$$

$$= \text{RHS}$$

Question 18

$$\tan 2x + \tan x = 0$$

$$\frac{2\tan x}{1-\tan^2 x} + \tan x = 0$$

$$\tan x\left(\frac{2}{1-\tan^2 x} + 1\right) = 0$$

$$\tan x = 0 \quad \text{or} \quad \frac{2}{1-\tan^2 x} + 1 = 0$$

$$x = 0°, 180°, 360° \qquad \frac{2}{1-\tan^2 x} = -1$$

$$-2 = 1 - \tan^2 x$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

$$x = 60°, 120°, 240°, 300°$$

$$x = 0°, 60°, 120°, 180°, 240°, 300°, 360°$$

Question 19

$$\begin{bmatrix} a & b \\ ka & kb \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ kax+kby \end{bmatrix}$$
$$= \begin{bmatrix} ax+by \\ k(ax+by) \end{bmatrix}$$

If $y = ax + by$, then $y = kx$

Question 20

a $\begin{bmatrix} 4 \\ 1 \end{bmatrix}\begin{bmatrix} -3 & 5 \end{bmatrix} = \begin{bmatrix} -12 & 20 \\ -3 & 5 \end{bmatrix}$

b $\begin{bmatrix} -3 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3\times4+5\times1 \end{bmatrix}$

$= \begin{bmatrix} -7 \end{bmatrix}$

Question 21

$w = 2 + i$

a $z = \overline{w} = 2 - i \Rightarrow$ Graph B

b $z = a - i \Rightarrow$ Graphs B and D

c $(2+i)(a+bi)$

$= 2a + ai + 2bi + bi^2$

$= 2a - b + (a+2b)i$

If $2a - b + (a+2b)i$ is real $a + 2b = 0 \Rightarrow a = -2b$

$\text{Re}(z) = -2\,\text{Im}(z)$

$\Rightarrow$ Graphs A, B and F

d $\text{Im}(w) = \text{Im}(z) = 1 \Rightarrow$ Graphs A and C

e $\mathrm{Im}(w) = |\mathrm{Im}(z)|$

Look for z such that $\mathrm{Im}(z) = \pm i$

$\Rightarrow$ Graphs A, B, C and D

f $|\mathrm{Im}(w)| = \mathrm{Im}(z)$

Look for z such that Im(z) = 1

$\Rightarrow$ Graphs A and C

g $z = iw$

$= i(2+i)$

$= 2i + i^2$

$= -1 + 2i$

$\Rightarrow$ Graph E

h $\dfrac{\overline{w}}{z} = \dfrac{2-i}{z}$

Graph A : $\dfrac{2-i}{-2+i} = \dfrac{2-i}{-(2-i)} = 1$

Graph B : $\dfrac{2-i}{2-i} = 1$

Graph C : $\dfrac{2-i}{i}$

Graph D : $\dfrac{2-i}{-2-i}$

Graph E : $\dfrac{2-i}{1+2i}$

Graph F : $\dfrac{2-i}{4-2i} = \dfrac{2-i}{2(2-i)} = \dfrac{1}{2}$

Graphs A, B and F

Question 22

a $$\begin{aligned} A^2 &= BCB^{-1}BCB^{-1} \\ &= BCCB^{-1} \\ &= BC^2B^{-1} \end{aligned}$$

b $$\begin{aligned} A^3 &= BCB^{-1}BCB^{-1}BCB^{-1} \\ &= BC^2B^{-1}BCB^{-1} \\ &= BC^2CB^{-1} \\ &= BC^3B^{-1} \end{aligned}$$

c $A^n = BC^nB^{-1}$

Question 23

RTP: $1\times2\times3+2\times3\times4+3\times4\times5+...+n(n+1)(n+2)=\frac{n}{4}(n+1)(n+2)(n+3)\ \ \exists n\in\mathbb{Z},n\geq1$

When $n=1$

LHS: = 1 × 2 × 3 = 6

RHS: $\frac{1}{4}\times2\times3\times4=6$

The statement is true for the initial case.

Assume it is true when $n=k$

$$1\times2\times3+2\times3\times4+3\times4\times5+...+k(k+1)(k+2)=\frac{k}{4}(k+1)(k+2)(k+3)\ \ \exists k\in\mathbb{Z},k\geq1$$

When $n=k+1$

$$1\times2\times3+2\times3\times4+3\times4\times5+...+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$

$$=\frac{k}{4}(k+1)(k+2)(k+3)\ \ +(k+1)(k+2)(k+3)$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$=(k+1)(k+2)(k+3)\,\frac{(k+4)}{4}$$

$$=\frac{(k+1)}{4}(k+2)(k+3)(k+4)$$

If the statement is true when $n=k,$ it is also true for $n=k+1.$

Given that is true when n = 1, by induction

$$1\times2\times3+2\times3\times4+3\times4\times5+...+n(n+1)(n+2)=\frac{n}{4}(n+1)(n+2)(n+3)\ \ \forall n\in\mathbb{Z},n\geq1$$

Question 24

RTP: $2^{n-1}+3^{2n+1}=7M,\ \forall n\in\mathbb{Z}, n\geq 1$

When $n=1$

$2^0+3^3=28$
28 is a multiple of 7.
The statement is true for the initial case.

Assume it is true when $n=k$

$2^{k-1}+3^{2k+1}=7M,\ \forall k\in\mathbb{Z}, k\geq 1$

When $n=k+1$

$$
\begin{aligned}
&2^n+3^{2(n+1)+1}\\
&=2^n+3^{2n+3}\\
&=2^n+9\times 3^{2n+1}\\
&=2^n+2\times 3^{2n+1}+7\times 3^{2n+1}\\
&=2^{-1}\times 2^n\times 2^1+2\times 3^{2n+1}+7\times 3^{2n+1}\\
&=2(2^n 2^{-1}+3^{2n+1})+7\times 3^{2n+1}\\
&=2\times 7M+7\times 3^{2n+1}\\
&=7(2M+3^{2n+1})\text{ which is a multiple of 7}
\end{aligned}
$$

If the statement is true when $n=k$, it is also true for $n=k+1$.

Given that is true when n = 1, by induction

$2^{n-1}+3^{2n+1}=7M,\ \forall n\in\mathbb{Z}, n\geq 1$

Question 25

RTP: $5^n + 3\times 9^n = 4M,\quad n \geq 0$

When $n = 0$

$5^0 + 3\times 9^0 = 4$
4 is a multiple of 4
The statement is true for the initial case

Assume it is true when $n = k$

$5^k + 3\times 9^k = 4M,\quad n \geq 0$

When $n = k+1$

$$
\begin{aligned}
&5^{k+1} + 3\times 9^{k+1} \\
&= 5\times 5^k + 3\times 9^n \times 9 \\
&= 5\times 5^k + 5\times 3\times 9^n + 4\times 3\times 9^n \\
&= 5(5^k + 3\times 9^n) + 4\times 3\times 9^n \\
&= 5\times 4M + 4\times 3\times 9^n \\
&= 4(5M + 3\times 9^n) \text{ which is a multiple of 4}
\end{aligned}
$$

If the statement is true when $n = k$, it is also true for $n = k+1$.

Given that is true when n = 0, by induction $5^n + 3\times 9^n = 4M,\quad n \geq 0$

Question 26

$$
\begin{aligned}
\text{LHS} &= \sin\theta(\sin\theta + \sin 2\theta) \\
&= \sin\theta(\sin\theta + 2\sin\theta\cos\theta) \\
&= \sin^2\theta + \sin\theta\ 2\sin\theta\cos\theta \\
&= 1 - \cos^2\theta + 2\sin^2\theta\cos\theta \\
&= 1 - \cos^2\theta + 2(1-\cos^2\theta)\cos\theta \\
&= 1 - \cos^2\theta + 2\cos\theta - 2\cos^3\theta \\
&= \text{RHS}
\end{aligned}
$$

Question 27

$$4\sin x\cos^2 x - \cos x = 0$$

$$\cos x(4\sin x\cos x - 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad 4\sin x\cos x - 1 = 0$$

$$x = \frac{\pi}{2} + \pi n, n \in \mathbb{Z} \qquad 2\sin 2x = 1$$

$$\sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12} + \pi n, n \in \mathbb{Z}$$

Question 28

a $$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b $$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

c $y = \frac{\sqrt{3}}{3}x;$

$$m = \tan\theta = \frac{\sqrt{3}}{3}$$

$$\theta = 60^\circ$$

$$\begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

d $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

If we let the square have vertices A (0, 0), B (0, 2), C (2, 2) and D (2,0), we find the coordinates after the two-stage transformation to be $A'(0,0), B'(\sqrt{3}, 1), C'(\sqrt{3}-1, \sqrt{3}+1), D'(-1, \sqrt{3})$

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} & \sqrt{3}-1 & -1 \\ 0 & 1 & \sqrt{3}+1 & \sqrt{3} \end{bmatrix}$$

The same square after the one 30° anticlockwise rotation has coordinates

$A'(0, 0), B'(-1, \sqrt{3}), C'(\sqrt{3}-1, \sqrt{3}+1), D'(\sqrt{3}, 1)$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \sqrt{3}-1 & \sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3}+1 & 1 \end{bmatrix}$$

This shows that while the vertices of the square have the same location, it is not the same image as the coordinates of vertex B and D are swapped.